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On frequency independent damping

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Abstract

The following three models, which lead to the frequency independent damping, are examined: the hysteretic model, which consists of linearly elastic spring and (in parallel with the spring) Coulomb's element with the force proportional to the value of displacement; the modified hysteretic model, which is constructed by the addition (sequential) of an elastic spring to the foregoing model; the quasi hysteretic model, which has a structure similar to the hysteretic model but takes account of mean values of displacements and velocities. Some basic problems of theory of vibrations for a single-degree-of-freedom system (action of an instantaneous impulse; free vibration due to an initial non-zero displacement; response to a suddenly applied constant force; vibration under the action of periodic forces) are considered. A comparison is carried out for results corresponding to the models and also to the model with constant complex stiffness and model with viscous damping.

1. Introduction

The concept of frequency independent damping arose in 1927 when Kimball and Lovell [1] discovered that many engineering materials exhibit a type of internal damping in which the energy loss per cycle is proportional to the square of the strain amplitude and is independent of the frequency of the applied sinusoidal strain. This result was confirmed further by many investigators. A mathematical description of the corresponding damping force was done by Theodorsen and Garrick [2]: "The structural friction can be described by a force in phase with the velocity but of a magnitude proportional to the restoring force. With each restoring force term, say αC_α, there will be a friction term $i\alpha g_\alpha C_\alpha$, in which g_α is the damping coefficient". It is implied that steady harmonic vibrations in the complex form $\exp(i\omega t)$ (ω is circular frequency, t is time)

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are considered. It is seen that the restoring force and damping force are assumed as linear (α is a deflection). Actually, the cited determination introduces the notion of the independent of frequency complex stiffness. A large number of papers (e.g., [3–17]) have been written on the subject of structural damping as it is determined in Ref. [2]. Other names for this kind of damping are hysteretic damping [6], ideal hysteretic damping [15], material damping [18] and rate-independent damping [19]. A circumstantial analysis of the concept can be found in Refs. [11–15]. An important result obtained independently by Fraeijs de Vuebeke [10], Caughey [11] and Crandall [13] is that the linear hysteretic damping violates the requirement of causality: the system responds before exiting. Also in Ref. [20] the analysis is given which concerns the transient response of the linear system with constant complex stiffness and the corresponding violation of the causality principle.

A spreading of the cited description given by Theodorsen and Garrick for the harmonic motion onto the general case of motion has been done by Reid [7]:

$$F = k\left(x + \eta|x|\frac{\dot{x}}{|\dot{x}|}\right), \tag{1}$$

where F is total force (linearly elastic plus damping) applied to the model, x is displacement, $\dot{x}$ is velocity (derivative by time), k and η are stiffness and damping coefficient, respectively. A hysteretic loop corresponding to a periodic motion with amplitude x_0 is shown in Fig. 1. The damping force in the form used in Eq. (1) is referred to in Ref. [21] as linear Coulomb friction force. For a harmonic motion in the form $x = x_0 \exp(\mathrm{i}\omega t)$, where x_0 is a positive amplitude of vibrations one obtains using Eq. (1) (for $\omega > 0$)

$$F = kx(1 + \mathrm{i}\eta). \tag{2}$$

The Eq. (2) exactly corresponds to the above cited structural friction description of Theodorsen and Garrick. This fact can lead to an erroneous opinion that model (1) and the model of structural (or linear hysteretic) damping are equivalent for steady harmonic motion [7,22]. However the application of the complex representation of forces and displacements has a meaning only for linear systems and is absolutely meaningless for a non-linear system with force of the

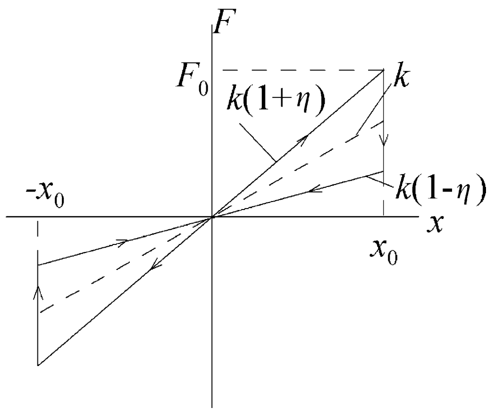

Fig. 1. Relation between force and displacement for the hysteretic model.

form (1), for which a possibility of separation of real or imaginary parts of the complex solutions in order to obtain a real solution does not exist. In fact, the steady state solution [12,21] for a one-degree-of-freedom system with the force (1), excited by a harmonic force $p_0 \cos(\omega t)$ or $p_0 \sin(\omega t)$, differs substantially from the corresponding solution for force (2) representing the linear hysteretic model. Note that in Ref. [12] the mentioned solution is obtained for non-stop motion and in Ref. [21] the one-stop motion in the half-period of vibration is permitted (in addition to force (1), also the viscous damping force and the constant Coulomb friction force are considered). The assumptions about the continuous or one-stop motion are justified if ω is large enough. Free vibration in the case of the force (1) are studied in Ref. [12]; the results of [12] relating to free vibrations were obtained again in Ref. [22].

In the paper the behaviour of model (1) in the cases not considered in previous investigations is studied: steady oscillations of a one-degree-of-freedom system under the action of a harmonic force without limitation on the number of stops, response to instantaneous application of a constant force to the system. This model will be called the hysteretic model. In parallel, two additional models are analyzed in the paper:

(a) modified hysteretic model obtained by sequential adding of an elastic spring to model (1), which eliminates discontinuities in the force (1) (when the velocity changes its sign) and removes the problem of stops,
(b) the quasi hysteretic non-linear model introduced in Ref. [23], which for steady vibration under the action of a harmonic force leads to the same results as the model (2) (from the structural damping description) and allows one to treat dynamic problems in time domain without difficulties.

The linear model corresponding to (2), which is called the complex stiffness model, and the model of equivalent viscous damping with the same energy loss per cycle at natural frequency as in the case of the considered non-linear models, are used in the paper for the purpose of comparison. Actually, the model of equivalent viscous damping is an appropriate choice if a mechanical system with one-degree-of-freedom is dealt with. For multi-degree-of-freedom systems, this model can be exploited for separate undamped modes of vibration [14,19], i.e., it is assumed that damping forces do not change the modes of vibration. However such a treatment is not appropriate if the system to be analyzed consists of parts with significantly different levels of damping, e.g., soil foundation and a structure interacting with soil [19]. In addition, in many cases when dealing with very large systems arising from the finite element method, it is desirable to carry out straightforward numerical integration in time domain without determination of natural vibration modes and frequencies for the considered system (this determination is inherent in the method of equivalent viscous damping). Complex stiffness model is widely used when studying steady harmonic vibrations. In other cases the principle of superposition can be applied assuming the linearity of the system; application of Fourier transform or series for obtaining a transient response of a mechanical system is also an application of the superposition principle. As mentioned above, such a treatment in the case of complex stiffness model leads to the violation of the requirement of causality. The three non-linear models which are the object of investigation in the paper are free of the mentioned flaws; they allow the numerical integration in time domain which leads to the transient or steady state response of the considered mechanical system.

2. Description of considered models

2.1. Hysteretic model

This model is determined by Eq. (1), and the corresponding hysteretic loop is shown in Fig. 1. When changing the sign of velocity, the force becomes discontinuous; the corresponding point (x, F) passes along a vertical line from the line with angle coefficient $k(1-\eta)$ to the line with angle coefficient $k(1+\eta)$ or vice versa. For eliminating some undesirable properties in the behaviour of the model, it will assumed throughout that $0 \leqslant \eta < 1$. The energy loss per cycle, W, is independent of frequency and equal to $2k\eta x_0^2$. Consider the damping ratio:

$$D = \frac{W}{2\pi k x_0^2} = \frac{\eta}{\pi}. \tag{3}$$

Note that together with this determination of damping ratio, another determination is used [23–26], in which instead of the value kx_0^2 in Eq. (3) the value $F_0 x_0$ appears (F_0 is the amplitude of force). Such a determination is suitable in the cases when it is difficult to separate out the potential energy of the considered system. When using the second determination of damping ratio, one obtains

$$D_1 = \frac{W}{2\pi F_0 x_0} = \frac{\eta}{\pi(1+\eta)}. \tag{4}$$

In what follows determination (3) will be used.

Although Eq. (1) is non-linear, it offers a property inherent in linear systems: if a function $x(t)$ entering (1) is changed by the function $Cx(t)$, where C is a constant, then the corresponding force will be equal to $CF(t)$. When studying vibrations of a mechanical system, this property leads to a possibility to compare results for model (1) with the corresponding results for linear systems.

2.2. Modified hysteretic model

The mechanical system leading to this model is shown in Fig. 2, where the part with parameters k_1 and η_1 corresponds to the hysteretic model determined by Eq. (1). Introducing an additional spring with the stiffness k_0 appears to be suitable, eliminating discontinuities in the force, which are inherent in model (1). Also the stops taking place in some cases of using model (1) are removed. The hysteretic loop for this model has the form shown in Fig. 3. The angle coefficients

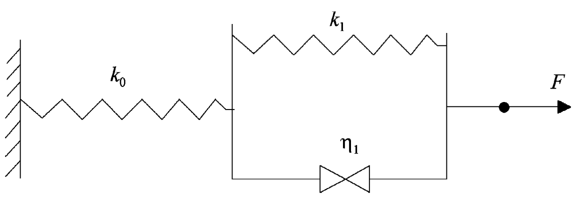

Fig. 2. Mechanical system leading to a modified hysteretic model.

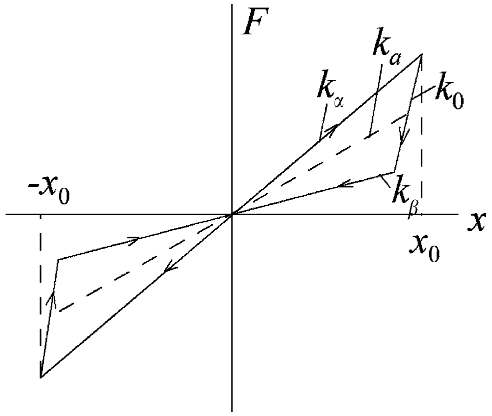

Fig. 3. Relation between force and displacement for the modified hysteretic model.

k_α, k_β, k_a in Fig. 3 are as follows:

$$k_\alpha = \left[\frac{1}{k_0} + \frac{1}{k_1(1+\eta_1)}\right]^{-1}, \quad k_\beta = \left[\frac{1}{k_0} + \frac{1}{k_1(1-\eta_1)}\right]^{-1}, \quad k_a = 0.5(k_\alpha + k_\beta). \tag{5}$$

If a point (x, F) lies at a moment on the line with angle coefficient k_α or k_β and the velocity changes its sign passing the zero value, then the point begins to move along the line with angle coefficient k_0, and the value F remains continuous. For time intervals in which the velocity does not change its sign, there are linear relations between Δx and ΔF with coefficients k_α, k_β or k_0. This description, following from the behaviour of the mechanical system shown in Fig. 2, is sufficient for obtaining a relation between the force and displacement, and can serve instead of a constitutive equation.

It is appropriate to deal with other independent parameters instead of k_0, k_1, η_1, namely

$$k_0, \quad k = k_a, \quad \eta = \frac{k_\alpha - k_\beta}{2k}. \tag{6}$$

This results in (similarly to the hysteretic model)

$$k_\alpha = k(1+\eta), \quad k_\beta = k(1-\eta). \tag{7}$$

Parameters k_1, η_1 (if needed) can be expressed through parameters k_0, k, η. According to Fig. 3, the energy loss per cycle will be

$$W = 2k\eta x_0^2 \frac{\beta - 1 - \eta}{\beta - 1 + \eta}, \tag{8}$$

where

$$\beta = \frac{k_0}{k}. \tag{9}$$

Using Eq. (3) the following relationship for damping ratio D is obtained

$$D = \frac{W}{2\pi k x_0^2} = \frac{\eta}{\pi}\frac{\beta - 1 - \eta}{\beta - 1 + \eta}. \tag{10}$$

Parameter η can be expressed through damping ratio:

$$\eta = r - \sqrt{r^2 - \pi D(\beta - 1)}, \tag{11}$$

where

$$r = 0.5(\beta - 1 - \pi D). \tag{12}$$

Note that the property of "linearity" mentioned in Section 2.1 takes place also for the modified hysteretic model. In fact, for the function $Cx(t)$, the plot on the plane (x, F) is similar (with the similarity centre in the origin of co-ordinates and with the coefficient of similarity C) to the original plot (for function $x(t)$). Thus the corresponding values of F will be C times as great as the force relating to $x(t)$.

2.3. Quasi hysteretic model

This model has been suggested in Ref. [23] and was used in non-linear seismic response analysis in Refs. [24,25]. The equation relating a force to a displacement has a structure similar to (1) but contains mean values (relative to time) of displacements and velocities:

$$F = k(x_m)\left[x + \eta(x_m)\dot{x}\frac{x_m}{\dot{x}_m}\right], \tag{13}$$

where $k(x_m)$ and $\eta(x_m)$ two functions depending on mean value of displacements; $\dot{x}_m$ is mean value of velocity (derivative $\dot{x}$). The weighted mean value is determined as follows:

$$z_m = \left[\frac{(n+1)}{t^{n+1}}\int_0^t s^n z^2(s)\,\mathrm{d}s\right]^{1/2}, \quad (z = x, \dot{x}). \tag{14}$$

As the parameter n is made greater than 0, correspondingly more relative weight is given to data close to time t. The name "quasi hysteretic" appears to be suitable since the velocity is included in constitutive equation (13), whereas the term "hysteretic" implies the independence of the velocity. Note that in the process of numerical solution of a system of differential equations which appears when using the model of the type (13) the corresponding integrals entering (14) are determined numerically. It is important that each time step adds a small portion to values of integrals already calculated by the beginning of the step. Using Newmark's method with iterations leads to effective solution of the corresponding non-linear system of differential equations [24,25].

It can be shown that in the case of harmonic variation ($z = z_0\sin(\omega t)$ or $z = z_0\cos(\omega t)$), the mean value, z_m when $t\to\infty$ tends to the constant $z_0/\sqrt{2}$ independently of n. For harmonic motion

$$x = x_0\sin(\omega t) \tag{15}$$

one obtains from Eq. (13) the relationship for steady state vibrations ($t\to\infty$):

$$F = k(x_0/\sqrt{2})[\sin(\omega t) + \eta(x_0/\sqrt{2})\cos(\omega t)]x_0. \tag{16}$$

The energy loss per cycle will be

$$W = \int_{t_0}^{t_0+2\pi/\omega} F(t)\dot{x}(t)\,\mathrm{d}t = \pi k(x_0/\sqrt{2})\eta(x_0/\sqrt{2})x_0^2. \tag{17}$$

If the behaviour of stiffness and damping depends on the amplitude of vibration then the functions $k(x_m)$ and $\eta(x_m)$ should be determined in accordance with this behaviour using Eqs. (16) and (17) (see Refs. [23–25]).

Further the case of constant values of k and η will be considered. In this case the relation (16) coincides with the relation following from the complex stiffness model (2) for steady vibration in the form (15) ($x = x_0 \exp(\mathrm{i}\omega t)$ is substituted into (2) and the imaginary part of the result is taken). Thus the quasi hysteretic model with constant k and η is equivalent to the complex stiffness model for the steady harmonic vibrations. Fig. 4a–c illustrate the process of approaching a limit elliptical loop for several values of parameter n (identical values of n are taken for x and $\dot{x}$). The convergence becomes slower with increasing n. Note that a family of models is obtained, each having the same steady behaviour in harmonic motion as the complex stiffness model. The values $n = 0$ or $n = 2$ will be used both for displacements x and velocities $\dot{x}$. According to Eqs. (17) and

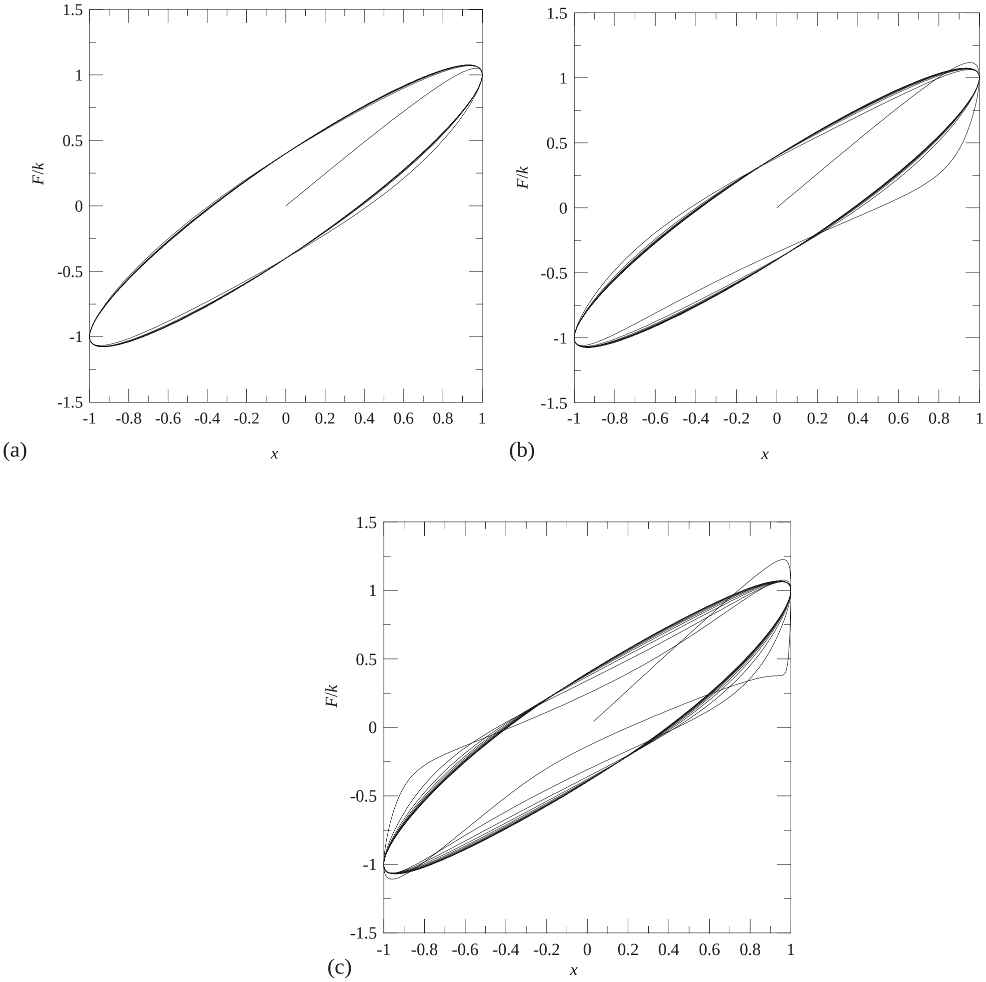

Fig. 4. Approaching the limit elliptic loop for quasi hysteretic model for $D = 0.2$ and different values of n: $n = 0$ (a), $n = 2$ (b), $n = 10$ (c).

(3) damping ratio for the model has the form

$$D = \frac{\eta}{2}. \tag{18}$$

The above mentioned property of "linearity" holds also for quasi hysteretic model.

3. Vibration of one-degree-of-freedom system

Consider the vibrations of a mass M attached to a "spring" which reacts with the force according to one of the three models described above; let an active force, $P(t)$, also act upon the mass. Introduce the non-dimensional time

$$\tau = t\sqrt{k/M} \tag{19}$$

and divide both sides of the equation of motion by k. The equation can be written in the form:

$$x'' + \tilde{F} = \tilde{P}(\tau), \tag{20}$$

where the designation of the second derivative of x by τ is used, and the tilde denotes that the corresponding forces are divided by k. For the hysteretic model and modified hysteretic model the whole time interval is divided into subintervals at which the Eq. (20) is linear and therefore a simple exact solution can be written for the subintervals (for an arbitrary function $\tilde{P}(t)$ the Duhamel's integral is needed). The moments of transition from one value of stiffness to another in $\tilde{F}$ are the moments in which the displacement x or derivative $\dot{x}$ pass through zero changing its sign; these moments are determined successively when going in time domain with a small step h (for τ). In the case of the hysteretic model value $\tilde{F}$ becomes discontinuous for moments when x' becomes zero and changes its sign and hence it is possible that x remains constant at some intervals of time (at these intervals, $F = P(t)$ and the point (x, F) in Fig. 1 lie at a vertical line between lines with angle coefficients $k(1-\eta)$ and $k(1+\eta)$). Note that for the value $\tilde{F}$ stiffnesses in Fig. 1 will be $1+\eta$ and $1-\eta$, and stiffnesses in Fig. 3 will be $1+\eta$, $1-\eta$ (instead of k_α and k_β, respectively), and β (instead of k_0). For a quasi hysteretic model it is appropriate to use Newmark's method (particularly, the method of mean accelerations) along with iterations executed at each time step. Let x_0, x'_0, x''_0 and x_h, x'_h, x''_h are displacement, velocity and acceleration at the beginning (the moment τ_0) and at the end (the moment $\tau_0 + h$) of a time step. They are related by the following equations:

$$x_h = x_0 + x'_0 h + 0.25(x''_0 + x''_h)h^2, \tag{21a}$$

$$x'_h = x'_0 + 0.5(x''_0 + x''_h)h, \tag{21b}$$

$$x''_h = \tilde{P}_h - x_h - \eta x'_h \frac{x_{mh}}{x'_{mh}}, \tag{21c}$$

where x_{mh}, x'_{mh} and $\tilde{P}_h$ are the mean value of displacements, mean value of derivatives (by τ) and the active force (divided by k) for the moment $\tau_0 + h$. The values with the zero index may be considered as known. In order to find the values with index h, it is supposed initially that $x''_h = x''_0$ (the predicted value), x_h and x'_h are found from (21a), (21b) and afterwards x_{mh} and x'_{mh} from definition (14). Doing so adds to the already known values of integrals in Eq. (14), corresponding to moment τ, additional small values, which depend on x_0, x_h, x'_0, x'_h; for this trapezium rule or Simpson's rule (introducing the point at $\tau_0 + 0.5h$) are appropriate. Further the corrected value of x''_h is found from (21c) and more accurate values of x_h and x'_h can be determined from (21a), (21b). This procedure, repeated two or three times, leads to sufficient accurate values of x''_h, x'_h, x_h and of integrals, appearing in the definition of mean values, for the moment $\tau_0 + h$. The accuracy is controlled by calculations with different values of the step of integration h. For multi mass systems such a method is also used; in this case, the forces acting upon masses from "springs" are defined by relative displacements of adjacent masses.

The property of partial linearity inherent in the considered models allows prediction of the response of the studied systems in proportionally changed conditions, that is if $x(t)$ is a solution corresponding to the initial values $x(0)$, $\dot{x}(0)$ and active force $P(t)$, then the function $Cx(t)$ (C is a constant) will be the solution for the initial values $Cx(0)$, $C\dot{x}(0)$ and active force $CP(t)$. For the hysteretic model (with the "Reid's spring") this statement has been pointed out in Ref. [12]. Thus calculations can be restricted in the case of free vibrations for $x(0) = 1$ ($x'(0) = 0$, $\tilde{P}(\tau) = 0$) or for $x'(0) = 1$ ($x(0) = 0$, $\tilde{P}(\tau) = 0$). In studying steady vibrations due to a periodic force $\tilde{P}(\tau)$ calculations can be carried out for the unit value of the amplitude of $\tilde{P}$; actually the dynamic response factor is defined in such a way.

For the purpose of comparison, the complex stiffness model (2) is considered and the model of viscous damping:

$$F = kx + \mu\dot{x}. \tag{22}$$

The damping parameter

$$\zeta = \frac{\mu}{2\sqrt{kM}} \tag{23}$$

is equal to damping ratio D at the natural circular frequency of vibrations

$$\omega_n = \sqrt{\frac{k}{M}}. \tag{24}$$

Taking $\zeta = D$, where D is the damping ratio defined above for the studied non-linear models, the model of viscous damping is obtained, which is called the equivalent viscous damping model. The models will have the same damping ratio at ω_n, for which the dissipative forces are most important.

3.1. Free vibrations

Results of calculation for the case $x(0) = 0$, $x'(0) = 1$ are shown in Fig. 5a,b. For $D = 0.05$ the three models lead to results very close to that corresponding to the equivalent viscous damping model, in the case $D = 0.2$ discrepancies in the results are more significant; note an increase in the period of free vibration (especially for the hysteretic model) compared to the viscous damping model. An analogous behaviour also takes place in the case of free vibration when $x(0) = 1$, $x'(0) = 0$ (Fig. 6a,b). Changing parameter n does not influence noticeably the response for quasi hysteretic models in the first case of motion, but effects significantly (for large values of D) this response in the case of the non-zero initial displacement: for $n = 0$, damping is more intensive than for $n = 2$ (see Fig. 6b). Fig. 7 illustrates the violation of the principle of superposition for

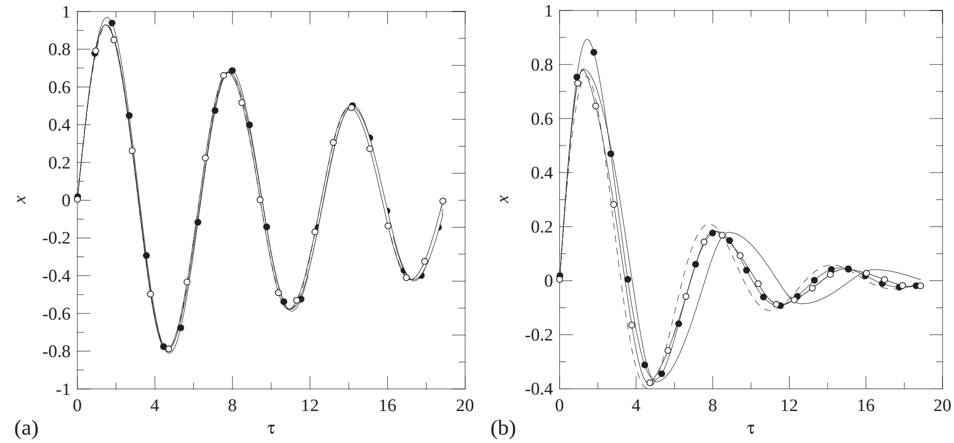

Fig. 5. Response of the models to initial velocity ($x(0) = 0$, $x'(0) = 1$) for $D = 0.05$ (a) and $D = 0.2$ (b). Keys: hysteretic model ——, modified hysteretic ($\beta = 50$) —○—, viscous damping - - - - -, quasi hysteretic ($n = 2$) —●—.

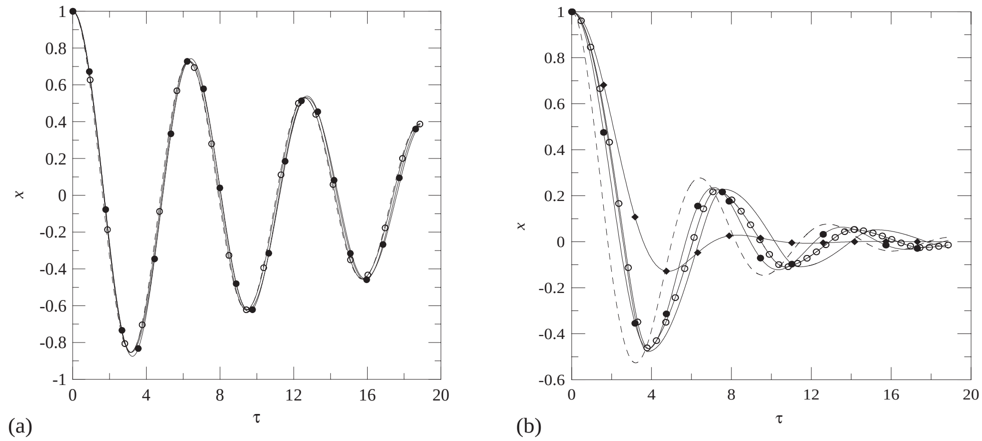

Fig. 6. Response of the models to initial displacement ($x(0) = 1$, $x'(0) = 0$) for $D = 0.05$ (a) and $D = 0.2$ (b). Keys: hysteretic model ——, modified hysteretic ($\beta = 50$) —○—, viscous damping - - - - -, quasi hysteretic ($n = 2$) —●—, quasi hysteretic ($n = 0$) —◆— (for b).

hysteretic model: the sum of two solutions (the dashed line in Fig. 7) corresponding to the initial conditions $x(0) = 0$, $x'(0) = 1$ and $x(0) = 1$, $x'(0) = 0$ differs from the solution for the initial condition $x(0) = 1$, $x'(0) = 1$ (the solid line). A similar behaviour is inherent in the remaining two non-linear models. The illustration (Fig. 7) is presented in order to underline once again the non-linearity of the model despite the opinion expressed in Ref. [22].

3.2. Action of constant force

Let a constant force $\tilde{P} = 1$ be suddenly applied to an one-degree-of-freedom system ($x(0) = 0$, $x'(0) = 0$). In this case, the hysteretic and modified hysteretic models exhibit a behaviour which

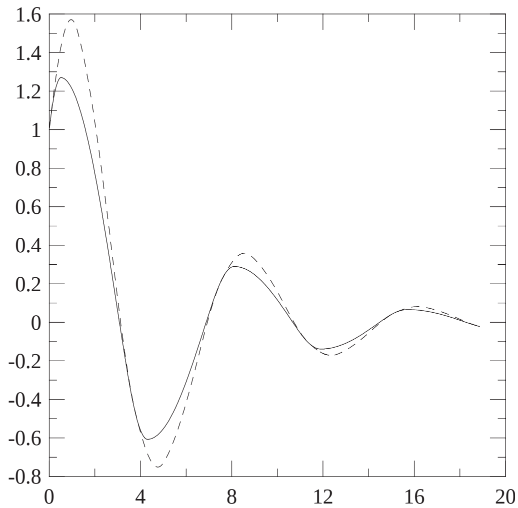

Fig. 7. Illustration of non-linearity of the hysteretic model for $D = 0.2$; solid line represents response to initial condition $x(0) = 1, x'(0) = 1$ and dashed line represents sum of two solutions for initial conditions $x(0) = 1$, $x'(0) = 0$ and $x(0) = 0$, $x'(0) = 1$.

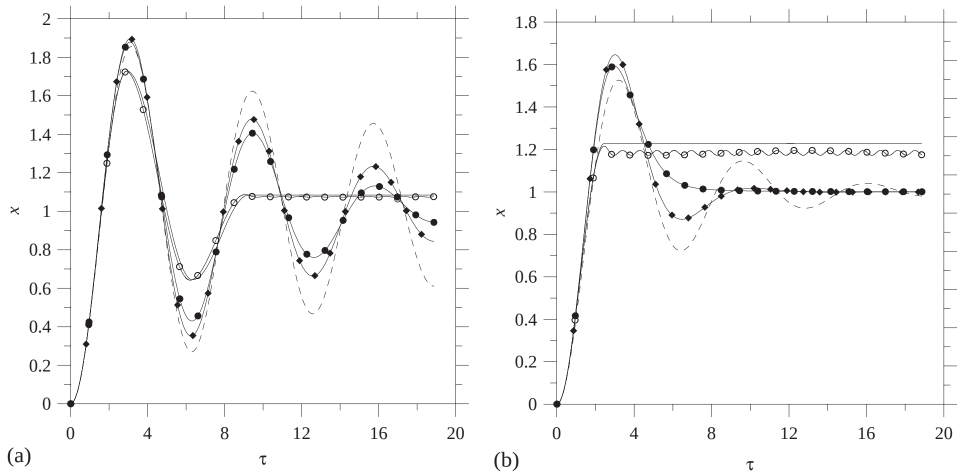

Fig. 8. Response of the models to instantaneous application of constant force $\tilde{P} = 1$ for $D = 0.05$ (a) and $D = 0.2$ (b). Keys: hysteretic model ——, modified hysteretic ($\beta = 50$) —○—, viscous damping - - - - -, quasi hysteretic ($n = 2$) —●—, quasi hysteretic ($n = 0$) —◆—.

differs significantly from that for the quasi hysteretic model and viscous damping model even for small damping (Fig. 8a,b). In the case of the hysteretic model, the motion stops at a time moment, at which the velocity becomes zero and the force corresponding to the low stiffness (i.e., $k(1 - \eta)$) is smaller than the active force or, alternatively, the active force is smaller than the force corresponding to the high stiffness (i.e., $k(1 + \eta)$). In the case of modified hysteretic model, high frequency oscillations (corresponding to stiffness k_0) take place instead of the stop; for smaller values of β the amplitude of these oscillations increases. Results for the quasi hysteretic model are similar to that for the viscous damping model. Note that now damping is greater for $n = 2$ due to more weight being given to displacements at an interval of time when they are large, whereas for $n = 0$ small displacements at an initial period of motion have a noticeable influence (see Eq. (14)), which leads to decreasing damping.

3.3. Steady vibration due to periodic force

Consider for the three studied models steady vibrations of one-degree-of-freedom system under the action of the harmonic force:

$$P(t) = P_0 \sin(\omega t) = P_0 \sin(\tilde{\omega}\tau), \tag{25}$$

where P_0 and ω are the amplitude of the force and its circular frequency, $\tilde{\omega}$ is non-dimensional frequency,

$$\tilde{\omega} = \frac{\omega}{\omega_n}. \tag{26}$$

The amplitude of variation of $\tilde{P}(\tau)$, i.e., P_0/k (static deflection), will be taken equal to 1 in the calculations, which means that dynamic response factor is determined. Remember that the quasi hysteretic model is equivalent to the constant complex stiffness model for the considered motion. For the hysteretic model and modified hysteretic model the steady state periodic motion with the period $2\pi/\tilde{\omega}$ (for τ) is obtained after a number of periods of the acting force, controlling the displacements and derivatives at the beginning and at the end of the periods. Displacements x corresponding to steady vibrations are shown for two values of $\tilde{\omega}$ in Fig. 9a,b (for $D = 0.05$) and Fig. 10a,b (for $D = 0.2$); τ is non-dimensional time reckoning from the beginning of the period of $\sin(\tilde{\omega}\tau)$, for which the motion can be considered as steady state. Note that for the steady vibration absolute values of maximum and minimum displacements are equal. It can be seen that for hysteretic model at $D = 0.05$, $\tilde{\omega} = 0.25$, the steady motion has two stops within the half-period; as

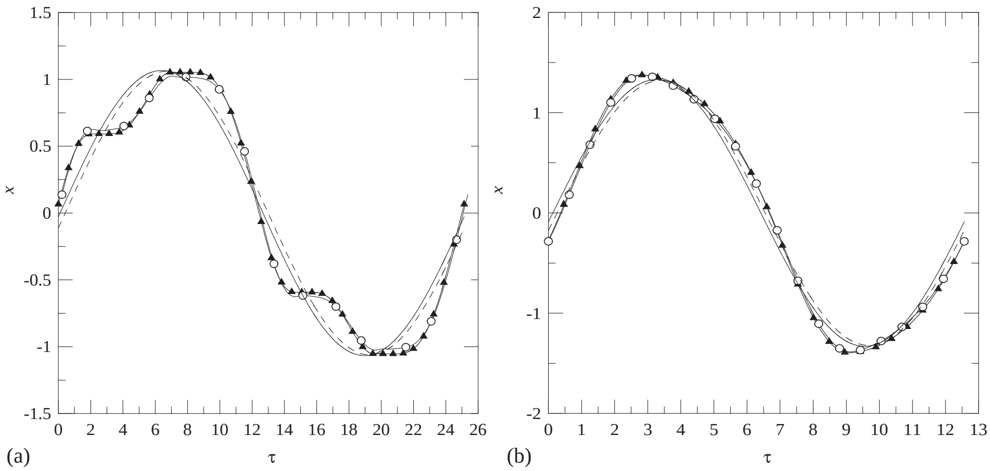

Fig. 9. Steady vibrations for two values of normalized frequency: $\tilde{\omega} = 0.25$ (a) and $\tilde{\omega} = 0.5$ (b), and damping ratio $D = 0.05$. Keys: complex stiffness and quasi hysteretic models ——, viscous damping - - - - -, hysteretic —▲—, modified hysteretic ($\beta = 50$) —○—.

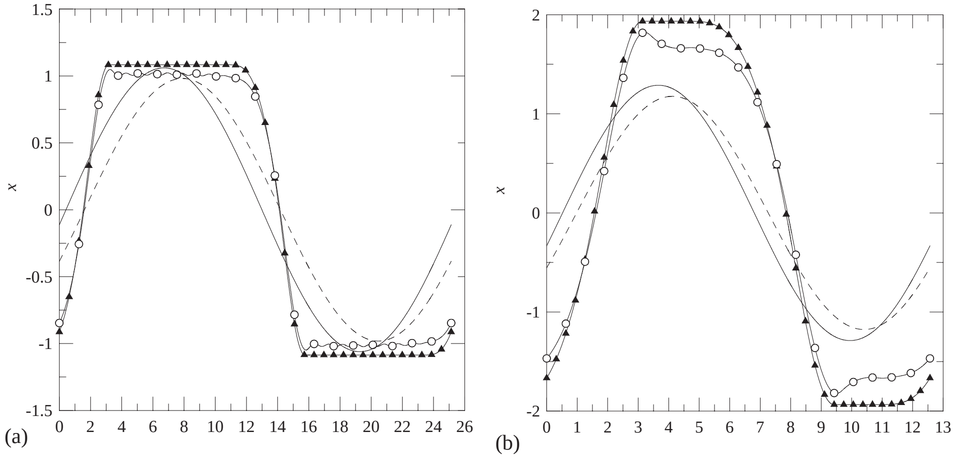

Fig. 10. Steady vibrations for two values of normalized frequency: $\tilde{\omega} = 0.25$ (a) and $\tilde{\omega} = 0.5$ (b), and damping ratio $D = 0.2$. Keys: complex stiffness and quasi hysteretic models ——, viscous damping - - - - -, hysteretic —▲—, modified hysteretic ($\beta = 50$) —○—.

calculations show for this value of damping, non-stop motion is realized if $\tilde{\omega} > 0.38$. In the case $D = 0.2$, non-stop motion takes place for $\tilde{\omega} > 0.68$ and one-stop motion is realized if $0.13 < \tilde{\omega} \leqslant 0.68$. For these values of frequency, the modified hysteretic model leads for the chosen value of β, to the results close to that for the hysteretic model. Even for a small enough value of D, as 0.05, the behaviour of the hysteretic model and modified hysteretic model can differ significantly from the remaining three models (Fig. 9a). The discrepancies become drastic for greater values of damping (Fig. 10 a,b). This produces limitations on possibility of linearization for the hysteretic and modified hysteretic models. Note some surprising results corresponding to $\tilde{\omega} = 0.5$: for the hysteretic and modified hysteretic models the amplitude of steady state vibrations at $D = 0.2$ is noticeably greater than the amplitude at $D = 0.05$ (Fig. 9b and Fig. 10b).

Fig. 11 a–c presents the dynamic response factor (i.e., the amplitude of vibration for unit value of the static deflection P_0/k) as a function of normalized frequency $\tilde{\omega}$. For $D = 0.05$, the results are closely related (excluding an interval of small values of $\tilde{\omega}$). For greater values of D, the hysteretic model and modified hysteretic model lead to results which differ significantly from those corresponding to the other three models.

Consider one example of a periodic non-harmonic excitation for a force equal P_0 during the first half of the period T_0 and equal $-P_0$ during the second half. The steady state periodic motion in the case of the complex stiffness model and viscous damping model is determined by the application of Fourier series, and in the case of the non-linear hysteretic models the above mentioned numerical integration scheme in time domain is used until a periodic response is achieved. Results of calculation for the unit value of P_0/k and $\tilde{\omega} = 0.25$ are shown in Fig. 12a ($D = 0.05$) and 12b ($D = 0.2$) (now $\tilde{\omega} = 2\pi/(T_0\omega_n)$). Again it can be seen that for the hysteretic model and modified hysteretic model increasing damping can lead to an increase in response. Note that in the case $D = 0.2$ results for quasi hysteretic model and complex stiffness model are noticeably different while solutions for a harmonic force are identical for these models. This is a consequence of non-linearity of the quasi hysteretic model.

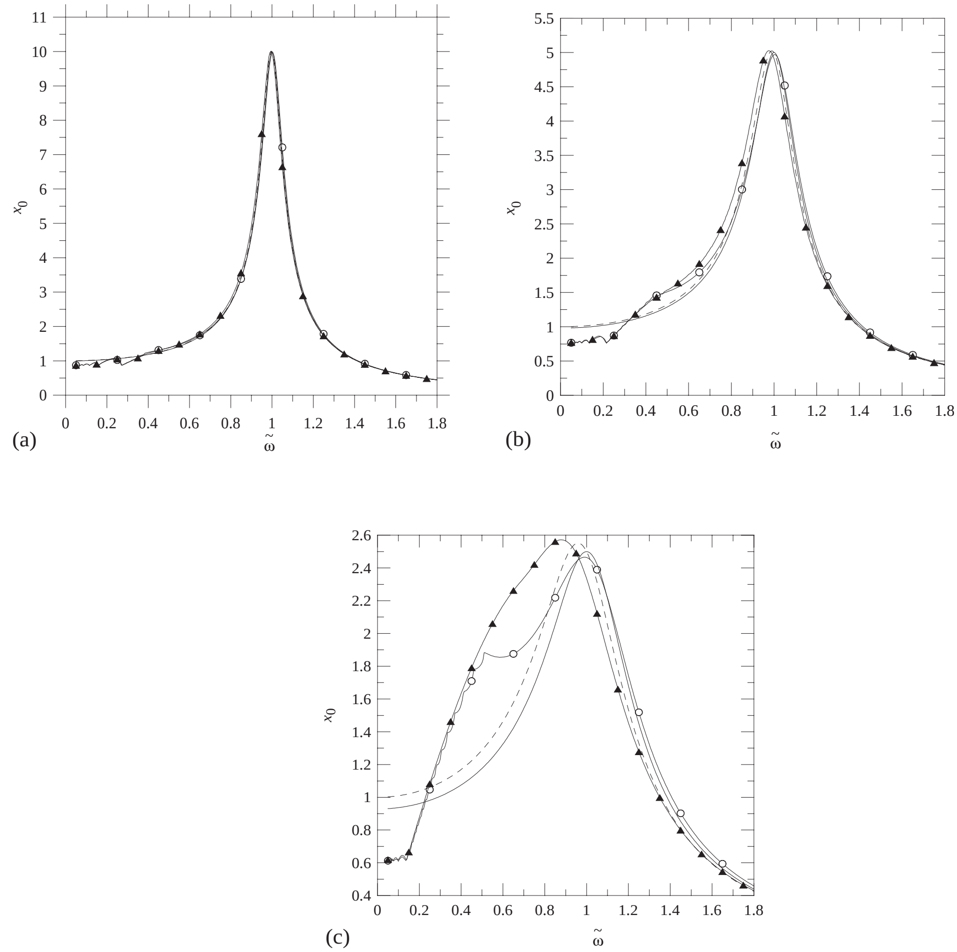

Fig. 11. Dynamic response factor for different models under harmonic exiting force for values of damping ratio $D = 0.05$ (a), $D = 0.1$ (b), $D = 0.2$ (c). Keys: complex stiffness and quasi hysteretic models ——, viscous damping - - - - -, hysteretic —▲—, modified hysteretic ($\beta = 50$) —○—.

4. Concluding remarks

The study of the three non-linear models, leading to the frequency independent damping, has been performed in order to construct an alternative for the two popular models—equivalent viscous damping model and constant complex stiffness model, which have some deficiencies when an analysis of transient vibrations of large multi-degree-of-freedom mechanical systems is needed. In most considered examples of motion, the behaviours of the models are closely related when damping ratio D is small. For large values of damping the results corresponding to the hysteretic model and modified hysteretic model can differ drastically from those corresponding to the quasi hysteretic model, equivalent viscous damping model and constant complex stiffness model. This is especially true for steady state vibrations under the action of a harmonic or periodic force, and for

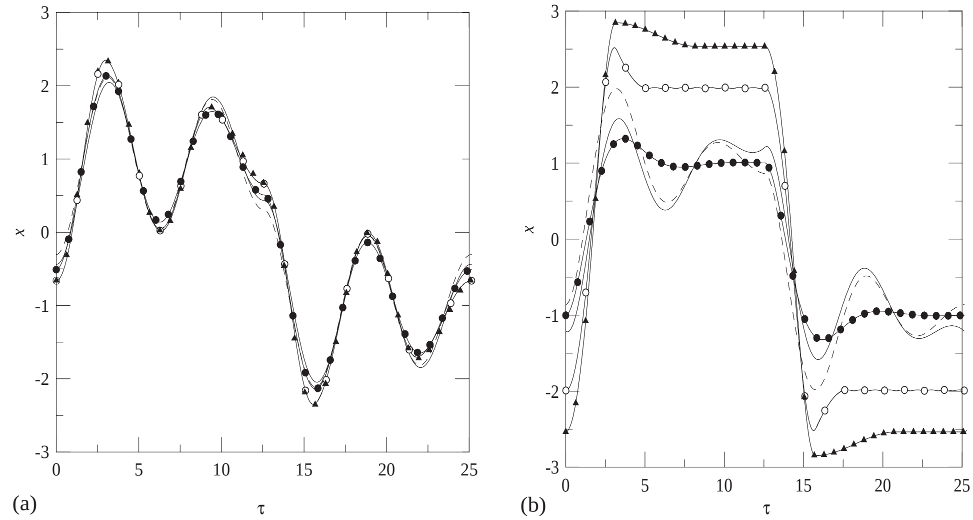

Fig. 12. Steady vibrations under periodic force for $\tilde{\omega} = 0.25$ and values of damping ratio $D = 0.05$ (a) and $D = 0.2$ (b). Keys: complex stiffness ——, quasi hysteretic models ($n = 0$) —●—, viscous damping - - - - -, hysteretic —▲—, modified hysteretic ($\beta = 50$) —○—.

transient vibrations due to a constant force when the hysteretic model leads to the stop of motion, and the modified hysteretic model leads to high frequency oscillations with displacements close to the displacement corresponding to the stop in the hysteretic model. The quasi hysteretic model dealing with mean values of displacements and velocities eliminates the deficiencies of the two remaining non-linear models. An attractive property of this model is the same response on the harmonic excitation as that for the constant stiffness model, which is considered to be acceptable for pure harmonic vibrations. Thus the quasi hysteretic model represents a suitable extension of the constant complex stiffness model to an arbitrary kind of motion.

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